Nested Simulation for Economic Capital

A common definition of an insurer’s economic capital requirements is based around a 1-year Value at Risk (VaR) metric. This defines capital requirements in terms of some tail percentile (typically the 99.5th percentile) of the market-consistent value of the insurer’s balance sheet in 1 year’s time. The problem of estimating such a metric naturally leads to the concept of nested simulation.

What is nested simulation?

Due to the number of risk drivers and potential complexity of the (joint) distribution of these risk drivers it is natural to adopt a simulation approach to estimate the tail percentile. This involves generating a large number of real-world scenarios for the risk drivers under the assumed joint distribution, valuing the balance sheet under each scenario, ranking the resulting scenarios according to their value and estimating the 99.5th percentile as the 99.5% worst scenario.

Though this calculation sounds simple in principle, it can be hard to implement in practice. The liability side of the insurer’s balance sheet often contains complex options and guarantees which are hard to value. The dependency of future guarantee cash flows on multiple risk drivers and their dependence on the entire path of these risk drivers naturally leads to a simulation based approach to valuation. This results in a so-called nested stochastic problem, with a large number of risk-neutral ‘inner’ scenarios (used for valuation) branching from each real-world ‘outer’ scenario for the risk drivers. Insurers typically use between 1,000 and 10,000 scenarios to value their balance sheets (though the size of statistical error around valuation estimates may still be surprisingly large even at the upper end of this range).

With 10,000 outer scenarios, the total number of scenarios in such a nested stochastic calculation is thus likely to be at least 10 million, with future liability cash flows being calculated in each of these scenarios. Another important point to note is that the scale of the problem doesn’t just depend on the total number of scenarios but also crucially depends on the amount of computational effort required to calculate liability cash flows in each scenario.

In practice, the insurer’s liabilities are represented by a number of representative policies, or ‘model points’, with the number of model points employed by insurers typically lying in the range 10,000 to 100,000. A typical

1. Of course, the use of a finite number of scenarios means that the resulting estimate will be subject to sampling error. We will discuss the estimation of this error and methods for refining the estimate in a future paper.
fully nested stochastic approach to 1-year VaR may thus involve calculation of 100,000 model points over 10 million scenarios, which is generally regarded as being computationally infeasible given current computer technology. Does this mean that nested stochastic techniques cannot be used at all? Or can we find ways of cutting down the total number of computations to be manageable given current technologies?

Reducing the amount of computational effort

The computational effort for a single valuation can be represented by a valuation grid of computational ‘units’, as indicated by the red squares in the diagram opposite. In estimation of 1-year VaR there will be one such valuation grid per outer scenario. The total computational effort thus scales with the number of red squares in this grid and the total number of outer scenarios. If we want to reduce this computational effort we therefore need to cut down the number of outer scenarios in which we need to calculate the valuation grid and/or the number of computational units in each valuation.

Methods for cutting down the number of outer scenarios requiring a full Monte Carlo valuation can be considered. For example, we might consider valuing at a small number of outer scenarios spanning the space of risk drivers and fitting a function (or interpolating) between these. Such ‘grid’ or ‘curve fitting’ approaches have been suggested in the calculation of VaR of asset portfolios – see for example Chishti, *Simulation of Fixed-income Portfolios Using Grids*, Algorithmics Research Quarterly, Vol. 2, No. 2, June 1999. However, the applicability of this method to insurance liability books may be questionable given the dependence of insurance liabilities on a relatively large number of risk drivers – in such cases the number of outer scenarios required to span this multi-dimensional space is likely to be large.

Rather than considering reducing the number of outer scenarios, in this note we will focus on methods for cutting down the number of such computational units in each valuation grid – either by decreasing the width of the grid, or its height, or by sparse distribution of units across the grid.

**Method 1: Reduce number of model points**

Actuaries have traditionally focused a lot of modelling effort on highly granular representation of their liabilities. Such granular modelling may be feasible if using a relatively small number of scenarios but, as we have discussed, creates problems when a large number of scenarios are required.

However, for the estimation of capital requirements, are 100,000 model points really necessary to summarise the liabilities of a typical insurer? In practice two model points may be sufficiently ‘similar’ (in the sense of producing similar cash flows) that we are really doubling up effort for a relatively small return. At what point does the marginal increase in accuracy by adding one more model point fail to justify the additional unit of computational effort required?

The idea here is to attempt to find a relatively small subset of model points which produce ‘similar’ cash flows to the full set of model points, thus cutting down the width of our valuation grid (see opposite).

The reader who is familiar with the concept of *Replicating Portfolios* may recognise that the representative subset of model points is essentially acting like a replicating portfolio for the full set of model points. However, this

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2 For the calculation of 1-year VaR, we are assuming that we only need to calculate values at one future point in time (at the end of 1 year). In other applications we may want to calculate values at many future points in time. For example, we may be interested in a ‘run off’ projection of a dynamic hedging strategy over multiple time periods. In this case we would require multiple valuation grids per outer scenario, thus increasing the computational burden even further. The methods discussed in this note apply in this more general case.

3 The term ‘grid’ here should not be confused with the ‘valuation grid’ used elsewhere in the note.
replicating portfolio can contain complex model points reflecting the actual management actions of insurance business, rather than just ‘vanilla’ assets (of the type traded in capital markets). Capturing such management actions can be critical in the estimation of economic capital. Indeed, our research indicates that attempting to replicate complex liabilities using vanilla assets alone can lead to significant errors in estimated capital requirements.

Unlike vanilla assets, these model points will typically require simulation techniques to value (rather than using analytic formulae) but this may still be computationally feasible if the total number of such model points is sufficiently small.

**Method 2: Reduce number of inner scenarios**

Rather than (or in addition to) reducing the number of model points, we can try and reduce the number of inner scenarios. Are 10,000 inner scenarios really necessary to perform an accurate valuation? Well, as discussed above the magnitude of the statistical error in the estimation of a value can be surprisingly large even using such a seemingly large number of scenarios. However, academics have devised some clever techniques for supplementing ‘brute force’ simulation which allow us to (potentially significantly) cut down the number of scenarios while retaining the same level of accuracy.

We will briefly describe two such techniques.

Firstly, **Variance Reduction Techniques** attempt to use information about the known analytic results to supplement simulation estimates and thus reduce their statistical error (or, more importantly in the current context, reduce the total number of scenarios to achieve a certain level of accuracy). In particular, we can try and find ‘similar’ assets to our liabilities (called control variates) which can be valued analytically and use known errors in the valuation of the control variate to adjust our liability value estimates. This method and other variance reduction techniques are described in B&H Research Report 2007-1236, “Variance Reduction & Martingale Tests”.

**Least Squares Monte Carlo (LSMC)** is an alternative technique which is often used in valuation of American options, where the nested stochastic problem also emerges. The technique cuts down the number of inner scenarios to an extremely small number (potentially a single scenario per outer scenario). Though each such valuation will be very inaccurate, the Least Squares Monte Carlo technique further refines the estimates by fitting a smooth function through these inaccurate valuations thus taking advantage of diversification of error across different outer scenarios. This technique is popular in valuation of American options and is beginning to find application in calculation of insurers’ capital requirements - see Cathcart & Morrison, “Variable annuity economic capital: the least squares Monte Carlo approach”, Life & Pensions, October 2009 and Bauer, Bergmann & Reuss, “Solvency II and Nested Simulations – a Least-Squares Monte Carlo Approach”, University of Ulm working paper 2009. Given the use of a smooth function to describe liability values, LSMC can be considered somewhat analogous to the ‘curve fitting’ approach discussed in the context of reducing the number of outer scenarios, the key difference being that LSMC uses a large number of outer scenarios and small number of inner scenarios (taking advantage of diversification across outer scenarios) while the ‘grid’ approach uses a small number of outer scenarios and large number of inner scenarios. Given the relatively large dimensionality of the risk driver space in the current application, we believe that LSMC is a more appropriate ‘curve fitting’ methodology.

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Method 3: Efficient distribution of inner scenarios across model points

Rather than trying to cut down the total size of the grid, we can try and cut down calculation to certain key points. Given potential similarity of different model points, it may be inefficient to pass the same scenarios through these – as discussed above to some extent this is really just duplicating effort. Rather it may be more efficient to pass different scenarios through different model points.

This idea has recently been suggested in the context of a similar nested simulation problem arising in the estimation of VaR for a bank’s derivative book - see Gordy and Juneja, “Nested simulations in portfolio risk measurement” Management Science, 2009, Forthcoming.

In practice, the applicability of this technique will depend on the specific liabilities being valued. For certain insurance liability types, the cash flows on particular model points depend on the cash flows of other model points. For example, this would be the case for a with profits book with a bonus rule dependent on the overall level of assets in the fund. In such cases all model points need to be valued using the same scenarios. Nevertheless the technique may be appropriate for other insurance liability types which can naturally be broken down into independent valuation problems e.g. variable annuities.

Summary

Estimation of insurers’ capital requirements using a 1-year Value at Risk (VaR) metric naturally leads to the concept of nested simulation, involving calculation of liability cash flows in a vast number of economic scenarios. A ‘brute force’ approach to this problem is thought to be infeasible given current technologies. However, we believe that there are a number of possible approaches to cutting down the number of computations required to estimate VaR accurately, either by cutting down the number of model points and/or the number of scenarios and/or being careful about how scenarios are distributed across model points. Adopting such approaches may make nested simulation a feasible approach to calculation of capital requirements after all.
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