A Least Squares Monte Carlo Approach to Liability Proxy Modelling and Capital Calculation

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Introduction

This note outlines a method which can be used for creating a function which approximates a multi-dimensional insurance liability through the use of Monte Carlo Simulation and Regression. The process, model choices, automation and validation are discussed in detail.

Although a liability proxy function has many applications within insurance risk management this note will discuss use of proxy modelling in the context of a Solvency II 1 year VaR capital calculation.
1 The nested stochastic problem

The Solvency II regime advises a 1 year 99.5% VaR of net assets measure of capital. Ideally, the value of net assets (assets less liabilities) should be read from the full probability distribution as shown in Exhibit 1.

Exhibit 1
Capital read from the full distribution of net assets

The liabilities in this calculation are valued using market consistent principles including the time value of options and guarantees. We assume in this note that we require a Monte Carlo simulation to value the liabilities. In order to ascertain the full distribution in 1 year for assets and liabilities we are faced with a nested stochastic problem.

We need to perform a 1 year real-world simulation with many thousands of scenarios. Within each of these scenarios we need to perform a risk neutral Monte Carlo simulation with thousands of scenarios to value the assets and liabilities.

The scenario requirement for such a simulation is therefore in the millions or hundreds of millions.

Exhibit 2
Nested stochastic simulation schematic

Due to the runtimes of most insurance asset-liability models this is unfeasible with the hardware commonly in use today.
This note outlines a method by which the nested stochastic problem can be solved. It involves the use of a proxy model which can replace the need to revalue the liabilities by Monte Carlo simulation in each of the real world scenarios.

In order for this solution to be feasible we need the proxy model to:

- Give an accurate representation of the liabilities
- Be demonstrably accurate to within pre-defined tolerances
- Be fast to evaluate
- Be easy to create and automatable on an ongoing basis
- Include all risks to which the liabilities are exposed

The sections below describe how the method can be implemented and how these criteria are met.
2 The Least Squares Monte Carlo method

The basis for the liability proxy method is the Least Squares Monte Carlo Method. Several authors\(^1\) have proposed similar methods in finance, particularly in the field of American option pricing which is an inherently nested stochastic problem.

In this note we outline a hybrid method, drawing ideas from the LSMC method and curve fitting approaches.

The basic idea for insurance capital calculation is that the liability value in 1 year’s time can be approximated by a function found by least squares regression. Once we have a function that approximates the liability value we can pass a large number of real-world scenarios through the function to calculate a capital requirement.

The outline of the process for a 1 year 99.5% VaR capital (Solvency II SCR) calculation is shown below:

- Define the form of the true liability function
  - Choose risk drivers
  - Define 1 year time horizon
- Create fitting scenarios
- Choose regression method and basis functions
- Find best sub-set and fit of basis functions
- Validate the fitted function
- Use fitted function with a set of real-world scenarios to get the distribution of liabilities and capital requirement

Although the steps can be reliably automated within an insurance company reporting process, some choices are available to the modeller and particular care must be taken during the implementation.

The following sections describe each step in more detail.

2.1 Choose risk drivers

We start with the assumption that there exists a true liability function that could be valued by Monte Carlo simulation at all possible points in many dimensions. We also assume for convenience that this liability function is continuous\(^2\),\(^3\). We will call this the “true liability function” which is unknown.

In the real-world, the liability value is a function of a very large number of variables. For example, our liabilities may be a function of the value of thousands of bond and equity holdings in our portfolio, the health and longevity prospects of millions of insured lives and many other factors. However, for valuation, it is common to apply some reduction in the dimensionality of our risks. We may assume that our liabilities are well approximated by broad equity and bond indices and a population mortality table. A similar dimensionality reduction exercise must be undertaken in a capital calculation although the choice of risk drivers may be different.


\(^1\) It is important to differentiate between the liability value function and payoff function. For example, a digital option has a discontinuous, stepped payoff function but a continuous value function. Almost all liabilities we would encounter in practice will have a continuous liability value.

\(^2\) More formally, we require that the payoff function is square-integrable which is a very general requirement and should be easily achieved.
The choice of risk drivers is an exercise in parsimony: we must find a small⁴ number of variables that adequately explain the risk characteristics of our liabilities.

We should be able to write down explicitly the arguments of the true liability function we are trying to approximate. For example:

\[ f(\text{Equity Value}, \text{Equity Volatility}, \text{Interest Rate Level}, \text{Mortality Rate} \ldots) \]

For each source of risk, we must specify the number of factors that describe the risk and the form of the factors. Some examples of choices that may be made for common risk types are described below:

### 2.1.1 Equity risk

In a capital calculation, it is common to use only a small number of risk drivers to explain the variation on the value of equities. For most purposes one will suffice. This may be the value of all equity holdings in one year's time or equivalently, the percentage change over the year.

In cases where it is important within a portfolio to differentiate between different classes of equity (for example local and foreign holdings) a more detailed model may be used by adding more factors. However, it is often possible to have a richer equity return model (both for the 1 year real-world and multi-year risk neutral model) while still using only a single equity risk factor representing the total holding value to describe the liability function.

**Equity dimensionality reduction example:**

Suppose we make a guarantee to policyholders on a pool of equity assets which contain five different underlying countries’ equity index holdings and four currency exposures.

In a nested stochastic model we project the real-world value of these nine variables over year 1 and calculate the value of our total equity holding. We perform some rebalancing of the equity portfolio and continue the projection of the assets using the full 9 risk factors in our risk neutral model.

Although there are nine distinct risk drivers at work, we may significantly reduce the dimensionality of the liability function by thinking carefully about the rebalancing dynamics.

Three different methods are described below:

1) We assume that the equity portfolio is always rebalanced to the same portfolio mix. In this case the only risk driver necessary for the liability function is the value of the portfolio at the end of the year without any approximation or loss of accuracy.

2) We assume that we cut our holding of foreign equity to reduce risk after an equity fall irrespective of which equity assets have lost money. In this case, again a single variable for the value of the total equity holdings is sufficient for the liability value function.

3) We assume that we cut the holding of foreign equity in response to their relative underperformance. In this case, we would need to split the equity portfolio value into local and overseas components. This still significantly reduces the dimensionality of the liability function nine variables to two and accurately retains the dynamics of the nested stochastic calculation.

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⁴ Small here is relative. The smaller the number of risk drivers the faster ALM runtimes will be. As a rule of thumb, less than 10 risk drivers for any particular liability portfolio is a good objective although this is dependent on specific features of the liabilities and appetite for ALM runtimes.
2.1.2 Interest rate risk

Insurances liabilities are sensitive to changes in any point on the risk free yield curve. We may say that each forward rate represents an individual risk driver – giving 100 or more potential interest rate risk drivers. However, this level of granularity adds very little to a liability valuation or capital calculation.

In practice, when performing a Monte Carlo valuation we model the movements of the entire yield using a small number of underlying stochastic factors. A similar reduction in the dimensionality of the yield curve is necessary for both the 1 year real-world model and the liability function.

There are a number of possible ways to achieve this dimensionality reduction. For example we may use a two factor short rate model to simulate possible real-world yield curves over the first year. The mechanics of this model mean that changes in the two underlying risk factors generate changes in the entire term structure over the year. In this framework, it is convenient to use the same two underlying risk factors as arguments for the liability function. Following this method will mean that we have a direct unambiguous mapping of interest rate risk, avoid translating real-world yield curves back into new factors for use in the liability function and do not lose any accuracy in the liability function specification.

This method of using common underlying factors in the real world simulation and liability function applies to other forms of interest rate risk model too.

Other common methods of modelling the real-world change in interest rates over the first year are Libor Market Model and Principal Component Analysis style methods. In both of these cases we isolate a small number of factors that describe changes in term structures over the year. Just like with short rate models, it is convenient and efficient to use the underlying factors of the model as arguments in the liability function.

2.1.3 Volatilities

In most cases where Monte Carlo models are employed liabilities values are highly sensitive to market-implied volatility. If this is the case we may also want to build implied volatility factors into our liability function.

Risk factor identification here follows a similar reasoning to interest rates. Although implied volatility has many dimensions and complexities, we summarize the risk to our liabilities in a small number of factors. It makes most sense to use the stochastic factors and / or parameters of our risk neutral models as the arguments for our liability function.

If a stochastic volatility model is used such as the Barrie & Hibbert SVJD model6, we have a natural method of stressing implied volatility. For example in this model we have a coupled stochastic system, where we have modelled both the asset price process $S_t$ and the variance value process $\nu_t$, such that:

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6 Barrie & Hibbert Note: Stochastic Volatility Jump Diffusion Calibration, Dynamics & Implementation; Lawson 2008
The single risk driver \( v \) evolves stochastically and leads to a change in the entire volatility surface. This is a natural choice as the driver of equity-implied volatility risk in the liability formula because the model can be used in both real-world and risk neutral modes\(^6\).

For deterministic volatility models such as the constant volatility model underlying the Black-Scholes formula, it is still possible to include implied volatility risk in the liability function. In this case we simply choose the fixed parameter as the volatility argument in our liability formula. If we include implied volatility in this way then we must also specify some consistent method of generating a distribution of implied volatility in the real-world. Simply adding an external correlated risk factor to a set of constant volatility real-world simulations (although not theoretically correct as in Heston model above) is a simple, robust method of including this source of risk in a capital calculation framework for most liability types.

Other methods may involve the use of both stochastic and fixed parameters in the stochastic volatility framework. For example, to generate broadly parallel shifts in the volatility surface using the Heston model we need to stress both the initial value of the stochastic volatility level \( \nu \) and its mean reversion level \( \theta \).

Similar considerations are required for interest rate implied volatility and other economic risks like credit spreads etc.

### 2.1.4 Non-market risks

In addition to market risk factors, insurance portfolios often have a number of non-market risk factors that affect the change in liability value over the year. For example, assumed mortality rates, assumed lapse rates and assumed take-up rates of policy options are all subject to uncertain change over the year and can have large effects on liability value and capital.

For these risk factors, we should use the same process, choosing a small number of risk factors that describe the dynamic of the portfolio. For example, we may apply a “percentage of table” scaling factor to the mortality rates \( q_x \) to describe broadly the changes in actuarial assumptions over the year.

Once we have chosen the set of risk factors that describe our liabilities, we need to choose our definition of the 1 year interval.

### 2.2 1 year definition

It is important to define exactly what we mean by 1 year.

The theoretically correct 1 year VaR capital is the amount of money required today so that we can be sure to be solvent in 1 year’s time, including the effect of all events that take place over the year: Our liabilities will reduce in duration; our assets and liabilities will have cash flows in and out; and changes in their composition; economic variables in the market will have experienced a full year of market risk and assumptions about non-economic risks such as mortality will evolve.

However due to the complexity of rolling forward ALM models for a 1 year time frame or for theoretical or internal consistency reasons, many insurance companies prefer to use a “Time 0 liability” approach. Here we assume that economic variables undergo a full year of market risk while “liability time” does not advance. This is the approach employed in the Solvency II standard formula.

\(^6\) B&H Note: 1-year real-world projection of equity implied volatility; Modelling and calibration methodology; Morrison 2011
There are many fine points which need to be considered under either of the liability time definitions, in particular, cash flows, rebalancing and management actions.

The important point to note here is that either of the time 1 or time 0 liability definitions can be used in the nested stochastic model which we will create the regression based fitting process to mirror.

The following charts illustrate the difference between the methodologies in a full nested stochastic calculation:

**Exhibit 3**
Nested stochastic simulation. Time 1 approach (left) and time 0 approach (right).

Other methods may be to stress all assets and liabilities over the first month or other time step or to employ a hybrid model where some variables like asset values are updated at time 0 whereas others like accounting variables are updated only at time 1.

Once we have decided the form and timing of the true liability function we begin the work of finding an approximation to it. The first step is to create a set of fitting scenarios.

### 2.3 Creating fitting scenarios

Fitting scenarios are used to create approximate liability valuations which will be used in fitting the proxy formula to the true liability value.

In creating fitting scenarios, we need first need to choose a multi-dimensional range over which we would like to approximate the true liability function. For each of the identified risk drivers we choose a range that contains all likely liability scenarios plus some extra allowance for what-if scenarios.

For example if we think the 99.5% worst case equity fall is 40% for a capital calculation we may want to fit the function to cover a 50% or 60% fall in equities for some safety. Further if we would also like to be able to estimate an updated capital requirement after a 99.5% event then we should try to fit our function to include at least a 64% fall in equities (40% capital after a 40% fall) or up to an 80% fall or more for safety.

An example multi-dimensional fitting range is shown below:

**Exhibit 4**
Example fitting range specification

<table>
<thead>
<tr>
<th>Variable</th>
<th>Min</th>
<th>Max</th>
<th>Initial Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity Value</td>
<td>0.1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Credit Spread</td>
<td>1%</td>
<td>20%</td>
<td>4%</td>
</tr>
<tr>
<td>Interest Rate Factor 1 Stress (Standard Deviations)</td>
<td>-6</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

….
Having chosen the fitting range, we create asset of 1 year, random\(^7\), multi-dimensional, uniform fitting scenarios. Each scenario contains a random change to the value of each of the chosen liability risk drivers over a 1 year term. We create a large number of these stressed positions. The more stresses we create, the more accurate our fitting will be. We call these “fitting outer scenarios”.

**Exhibit 5**
Fitting outer scenarios in two dimensions

![Fitting Outer Scenarios](image)

The use of fitting scenarios in this manner (proposed\(^8\) as an improvement to the Longstaff & Schwartz approach) allows us to fit over the entire range of interest with evenly distributed fitting scenarios. By contrast, a more traditional application\(^9\) of the least squares Monte Carlo technique would be based on a set of traditional nested stochastic scenarios (differing only by the small number of inner scenarios employed). Under this method our outer fitting scenarios would be the 1 year real-world paths. This method is much slower to converge in a capital calculation because of the relatively small number of fitting points in the tails of the distribution which are of most importance for a capital calculation. It also gives us no knowledge of the behaviour of the liabilities in un-sampled areas of the range of interest.

**Exhibit 6**
Real-world 1 year scenarios in two dimensions

![Real World Scenarios](image)

\(^7\) In fact, low discrepancy quasi-random numbers work better than pseudo random numbers for the fitting scenarios leading to faster convergence and improved stability of the algorithm and are easy to apply in practice.

\(^8\) N. Søndergaard Rasmussen (November 2002), Improving the least-square Monte-Carlo approach.

\(^9\) Barrie & Hibbert Note: Variable annuity economic capital: the least-squares Monte Carlo approach; Cathcart & Morrison
Once we have created these stressed fitting positions, we re-calibrate our risk neutral economic scenario generator using the stressed risk drivers, for example, using a new initial yield curve, equity value, volatility calibration etc. We then run a small number of random scenarios of our risk neutral economic scenario generator for each of the fitting positions. We call these “fitting inner scenarios”. Typically for a fixed scenario budget we aim to run a very large number of outer scenarios and a very small number of inner scenarios. The number of inner scenarios may be as low as 1 per outer scenario and must be independent of each other. Typical run sizes are in the region of 10,000 (outer) sets of two (inner) scenarios making 20,000 total ALM scenarios required. The rational for using two inner scenarios is that by using the antithetic sampling variance reduction technique we can have a relatively large reduction in the variability of the inaccurate valuations and therefore in the subsequent fitting.

Visualize the fitting scenarios required to run a 4 year projection in Exhibit 1.

Exhibit 7
Example fitting scenarios with two risk drivers over a 4 year liability term
Inner and Outer fitting scenarios using time 1 definition (left) and time 0 definition (right)
Fitting outer scenarios in green, fitting inner (risk neutral) scenarios in blue with 1 inner per outer scenario

Process manager software and the Barrie & Hibbert ESG API may be used to generate the fitting scenarios for this task in an automated robust manner. These sets of scenarios will be combined in a small number of grouped files for a bulk ALM model run.

Non-economic scenarios are added in a similar way. These may be stochastic over the first time step and deterministic thereafter or stochastic in both.

Once these fitting economic scenarios are run they must be run through the ALM software and the value of liabilities determined. In particular, for each outer fitting simulation, we must retrieve from the ALM model the average of the discounted liability values over all inner simulations. We discount to time 0 or time 1 depending on our definition of our 1 year time horizon specified earlier.

After completing this process, we should have a large number of inaccurate liability valuations (equal in number to the number of fitting outer scenarios). These valuations are correct on average but each one is inaccurate due to a large amount of random sampling error caused by the small number of inner simulations used. This effect can be seen when the inaccurate liability valuations are plotted against the risk drivers that generated them. An example for a European put option with one source of risk (equity value) is shown in Exhibit 8.
Exhibit 8
Results of fitting scenarios – inaccurate European put option valuations.
2000 outer scenarios with two antithetic inner scenarios each (4000 ALM scenarios total).

As we expect, the valuations are very inaccurate due to random error and at low equity values the valuations are usually higher and vice versa. We will see later how these are used to approximate the liability function.

An important consideration in generating these fitting scenarios is our exact definition of the liability and how we can aggregate different liability components.

2.4 Fitting liability data
To this point we have assumed that we are performing the exercise on a single liability book where we require a single liability value to be returned. In practice, a capital calculation must usually be undertaken on multiple liability books simultaneously and each book may have multiple components which comprise the net liability value.

If this is the case, we may wish to identify additive sub-components of the total liability and fit each individually. For example, an insurance portfolio may comprise a liability due to living benefits and one due to death benefits. We may further wish to add this portfolio to the value of with profits portfolio in the same organization. As long as we would simply add the individual components in a nested stochastic calculation, it is possible to fit a liability function for each one individually and create a total liability function by adding the individual functions. Alternatively, we may prefer to add the liability values in the scenarios first and fit a single liability function at the end.

Having access to more granular functions is beneficial for analysing and decomposing the components risk and capital. Doing this will also permit more advanced modelling of organizational structure and capital flows etc.

2.5 Choice of basis functions
Once we have our fitting data we can begin the fitting process. We will approximate the liability function by performing a regression through our inaccurate valuations.

A large body of statistical literature has built up over many years which describe various regression methodologies. We will describe just one of many possible methods in this note – polynomial regression.

We choose our regression basis functions to be a set of polynomial terms up to a pre-specified order in each of the risk drivers and their cross terms.

For example, in 2 risk dimensions a polynomial of degree 3 which may describe the liabilities is:
Where $L$ is the liability value, $R_1$ and $R_2$ are risk drivers and $a_1$ to $a_9$ are coefficients to be estimated.

We then perform an ordinary least squares regression of the inaccurate liability valuations on the basis functions. The regression coefficients calibrate the liability proxy function to the true liability function by smoothing through the random errors in the inaccurate liability valuations.

This is shown for the European put option example in Exhibit 9.

Exhibit 9
Regression fit of degree 4 polynomial to European put option

In fact, instead of ordinary polynomials, we use Legendre polynomials for the fitting. This class of function is beneficial for their orthogonal properties which promote numerical stability.

Polynomial functions are a very flexible tool for fitting many different types of curves and have performed well empirically on many liability types and in high risk dimensions.

Although the fitting is subject to a degree of sampling error from the randomness of the fitting scenarios, its convergence has been formally proved. As we add more scenarios and basis functions we are guaranteed to converge to the true liability function. In practice, the rate of convergence is dependent on the number of risk dimensions used, the variability of the inaccurate liability estimates and the smoothness and differentiability of the true liability function we are trying to approximate.

Techniques also exist to estimate the potential sampling error in a fitted function and therefore calculate the potential impact on estimated capital required and increase the number of scenarios used if needed.

A drawback of the polynomial regression technique is the large number of potential terms in the basis function. For even moderate numbers of risk drivers this can quickly increase into the thousands and may be greater than the number of fitting scenarios. In this case, a method is needed to select a subset of the polynomial terms that describes the liability function well without over fitting to the random liability valuations. The Akaike Information Criterion is a robust statistical test for this use. It compares competing sub-models, rewarding terms which increase descriptive power and penalizing the inclusion of terms which do not.

10 L. Stentoft, Convergence of the Least Squares Monte Carlo Approach to American Option Valuation, Management Science, Vol. 50, No. 9 (Sep., 2004),
11 Assuming some very general preconditions are met for the liability payoffs
In this way a variable selection algorithm such as a stepwise procedure or other method can be implemented which selects a parsimonious subset of terms for use in the regression. This allows us to use a fully automated process to generate the liability function which does not require excessive human intervention or judgement.

2.6 Validation

There are two main methods by which the fitted function may be validated.

The first involves Monte Carlo valuation of the true liability value at a number of important points. For example, we may choose a small number of stressed values at which we wish to validate the function. We can then run a large number of risk neutral (inner) simulations through the ALM model to more accurately value the liabilities. If these valuations agree (within some error bands) to the value predicted by the liability function then we can be fairly certain that our function will be universally accurate. The more points we use and the better we select these points, the more certain we can become.

The second method is to estimate standard errors bars around the fitted function. To do this we can use a “bootstrapping” technique. By re-fitting the liability function many times using sub-sets of the original data, we create a set of liability function estimates that create a confidence band around the original fit. We can calculate our capital using each of the sub-set fits which will give us an estimate of the potential standard error of the calculated capital. As an alternative to bootstrapping, we may use repeated random LSMC runs with new fitting scenarios. Although this approach takes longer to run, the sampling error characteristics of the liabilities should not change significantly from one valuation period to the next (if the same model and fitting range are used). We should therefore be able to conduct a formal fitting error experiment very infrequently (say once a year in a quiet modelling period) and use this to inform the formal reporting results.

2.7 Finding the real-world distribution of liabilities and calculating capital

Once we have validated the liability function, we can calculate our capital by passing a set of real-world risk driver values through the liability function. This produces a full distribution of possible liability values from which the VaR can be read. Typically a large number of real-world scenarios should be used for a VaR calculation to eliminate sampling error in the tail of the distribution.

3 Benefits of the LSMC method

We have seen above the process by which a least squares Monte Carlo proxy model can be created and used. We summarize the benefits below:

- The method uses relatively few scenarios (in the region of 20,000 compared to the many millions required for a full nested stochastic calculation)
- The method has a formal mathematical basis for convergence
- The choice of fitting points is done methodically and does not use human intuition or require us to trade off accuracy in different regions of the fit
- The choice of basis functions and fitting of the function can be completely automated
- Polynomial function are very flexible and able to fit to very efficiently to many different function shapes
- The fitting is very fast to perform (generally a few minutes to one hour for most practical applications)
- Evaluation of the proxy function is extremely fast
- All market and non-market risks and their joint behaviour can be modelled
- A confidence interval for the liability function and capital requirement can be estimated
- The proxy function and method for fitting and confidence can be easily explained to senior management and used within other parts of the business such as asset allocation, hedging etc
- The technique allows us to use a very large number of real-world simulations in the capital calculation, eliminating sampling error in the tail of the real-world distribution
Given these benefits, we believe that the method outlined above is a robust accurate technique for calculating and reporting a capital requirement.

4 Case study

The following charts show the results of a case study on a full ALM model conducted in conjunction with a large European insurer. The case study constructs a liability function by regression to approximate the value of a with profits liability portfolio.

The fitting is performed in 8 risk dimensions: two interest rate factors, interest rate volatility, equity value, equity volatility, credit spreads, lapse rates and mortality rates.

The fitting uses 20,000 total ALM scenarios for the fitting. The charts below show a validation of the fit, comparing the liability proxy function to more accurate Monte Carlo valuations using 1000 scenarios each.

Exhibit 10
Validation of liability function
As well as validating well in each individual dimension, we test how well the function predicts changes in simultaneous dimensions.

**Exhibit 11**
Validation of a number of joint stresses

We can see here that the function is an accurate description of the liabilities for stresses where multiple risk drivers are altered as well as for the single risk case.

By plotting the function in joint dimensions, we can gain insight into the joint behaviour of our liabilities. Exhibit 12 shows how our liabilities react to simultaneous changes in interest rates and lapse rates.
5 Control variates and replicating portfolios

It is also possible to combine the LSMC technique with a replicating portfolio (RP) method. More formally, in this case we treat a replicating portfolio as a control variate.

The objective of the RP is to explain a high percentage of the future cash flow variation. We then fit the liability function to the residual present value instead of the total present value as before. Our liability proxy is then the value of the control variate (or RP) plus the value of the fitted function.

If we can find a portfolio that explains a high percentage of the variation of the liability present values, then our fitting scenarios will have much less variability and we will be able to get more accurate fits with fewer scenarios required than before.

The benefits to combining these techniques are numerous:

- We have an alternative rigorous method to test the replicating portfolio
- We can correct for poor fitting in the RP
- We can adjust the RP to include non-market risks or risks not included in the RP fitting assets
- We achieve faster convergence in the fitting of the regression function

The following charts show how replicating portfolio and Least Squares Monte Carlo techniques can be used together.

We look at the fitting of a geometric Asian put option liability in 4 dimensions of risk: equity value, interest rate, equity volatility and strike. We assume that the strike price is a stochastic variable over the first year.
This assumption mimics a liability where the lapse rate, mortality rate, bonus rate etc create uncertainty in the value of the guarantee.

We first try to replicate this liability using only European put options. We use a range of strikes but assume that the strike in all replicating assets is fixed (rather than variable as in the liability).

After fitting, we calculate the exact value of the liability under a number of stresses and compare to the proxy model predicted value.

**Exhibit 13**
Fit to liability using replicating portfolio (Blue = true value, red = proxy model predicted value, green = difference)

Fit of RP under changes to equity and interest rate risks:

Fit of RP under changes to equity value and strike:
We can see from the above charts that the RP does a reasonable job of approximating the equity and interest rate risk of the liability but is a very bad predictor the liability value under changes to the strike. This is to be expected because of the assets we have selected.

Now, fitting a function to the difference between the RP and liability present values, and using this function to correct the RP predicted value, we can see that our combined proxy is a much better predictor of the liability value in all risk dimensions:

**Exhibit 13**
Fit to liability using replicating portfolio + LSMC function
(Blue = true value, red = proxy model predicted value, green = difference)

Fit of RP + LSMC under changes to equity and interest rate risks:
Fit of RP + LSMC under changes to equity value and strike:

By combining the techniques, we are able to achieve excellent results in all risk dimensions (better than either technique alone) without compromising on accuracy or ignoring material risks.

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